ELSEVIER

Journal of Pure and Applied Algebra 122 (1997) 243–250

JOURNAL OF
PURE AND
APPLIED ALGEBRA

Tight closure and strong test ideals

Craig Huneke [*,1]

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA

Communicated by T. Hibi; received 26 June 1996

Dedicated to the memory of Professor Hideyuki Matsumura

Abstract

For certain classes of rings we give an affirmative answer to whether there exists a uniform bound on the least power of the tight closure of an arbitrary ideal which lies in the ideal. The rings need to have the property that modulo each minimal prime there exists a resolution of singularities obtained by blowing up an ideal. In this case we prove the existence of a 'strong' test ideal, and then apply this existence to give an affirmative answer to the uniform bound question. © 1997 Elsevier Science B.V.

AMS Classification: 13A35; 13D25

1. Introduction

This paper is devoted to answering a question concerning elements in the tight closure of ideals and the degrees of the integral equations they satisfy. This question grew out of work on the uniform Artin–Rees theorem [8]. Essentially, we ask whether there is a uniform bound on the degrees of the integral equations satisfied by elements in the tight closure of ideals. Several questions for further study arise naturally from this work. The proof of the main result rests on the existence of what we call a 'strong' test ideal. We will first review the basic definitions and give a precise statement of the main problem. Our notation and basic definitions are taken from Matsumura's classic book [10].

The theory of tight closure is developed for any ring containing a field. The definition in equicharacteristic 0 is given by reduction to characteristic p. We will be applying

* E-mail: huneke@math.purdue.edu.

[1] The author was partially supported by a grant from the National Science Foundation.

PII S0022-4049(97)00053-4

tight closure in both equicharacteristic 0 and characteristic p in this paper. We first give the definition in positive characteristic.

Define R^{o} to be the complement of the union of all minimal primes of a ring R. The definition of tight closure for ideals in characteristic p is:

Definition 1.1. Let R be a Noetherian ring of characteristic $p > 0$. Let I be an ideal of R. An element $x \in R$ is said to be in the tight closure of I if there exists an element $c \in R^{\mathrm{o}}$ such that for all large $q = p^{\mathrm{e}}$, $cx^q \in I^{[q]}$, where $I^{[q]}$ is the ideal generated by the qth powers of all elements of I.

There are several definitions of tight closure in equicharacteristic 0. In this paper we will use the equational tight closure.

Definition 1.2. Let R be a locally excellent ring containing a field of characteristic zero. An element z is in the tight closure I^* of an ideal I if there exists a finitely generated $\mathbb{Z}$-subalgebra $R_{\mathbb{Z}}$ of R containing z such that the following holds: for all but finitely many closed fibers $\bar{R} = R_{\mathbb{Z}} \otimes \mathbb{Z}/p\mathbb{Z}$ of $\mathbb{Z} \to \mathbb{R}_{\mathbb{Z}}$, the image $\bar{z}$ of z in $\bar{R}$ is in the (characteristic p) tight closure of the image $\bar{I}$ of $I \cap R_{\mathbb{Z}} \subset R_{\mathbb{Z}}$ in $\bar{R}$.

This type of tight closure is also called $\mathbb{Q}$-tight closure, or the equational tight closure; see [3–5, 7].

We need several facts about tight closure.

Proposition 1.3. *Let R be an equicharacteristic Noetherian ring, and let I be an ideal of R. Then*

(1) *$I^* \subseteq \bar{I}$, the integral closure of I.*

(2) *An element $x \in R$ is in I^* iff the image of x in R/P is in the tight closure of the image of I in R/P for all minimal primes P of R.*

Proof. For statement (1) in positive characteristic, see [2, (5.2)]. In equicharacteristic 0, see [5, (4.1m)]. For the proof of (2) in positive characteristic, see [2, (6.25)]. In equicharacteristic 0, see [5, (6.3)]. □

We recall the definition of the integral closure.

Definition 1.4. The *integral closure* of I in a ring R is the set of all elements $x \in R$ such that x satisfies an equation of the form

$$x^k + a_1 x^{k-1} + \cdots + a_k = 0, \tag{1}$$

where $a_i \in I^i$. The integral closure of I is denoted $\bar{I}$.

An alternative definition for integral closure which makes it clear in characteristic p as why the tight closure is in the integral closure, is the following:

Given an ideal I in a Noetherian ring R, an element x is in the integral closure of I if there exists an element $c \in R^{o}$, the complement of the minimal primes of R, such that $cx^n \in I^n$ for infinitely many n (equivalently for all $n \gg 0$).

The least degree of a polynomial as in (1) which shows that x is integral over I, is unbounded if one varies I through all ideals of a Noetherian ring R as long as the dimension of R is at least 2. Let (R,m) be a Noetherian local ring of dimension at least 2. Let $x_1,\dots,x_d$ be a system of parameters. The element $u = x_1^{n-1}x_2$ is integral over the ideal $I = (x_1^n,\dots,x_d^n)$, but the least degree polynomial which u satisfies showing it is integral over I (as in (1)) is n, namely $u^n - (x_1^n)^{n-1}(x_2^n) = 0$. To see that no smaller power is possible, first complete R and go modulo a minimal prime above $(x_3,\dots,x_d)$. We obtain a two-dimensional complete domain. We may then replace this ring by its integral closure. It then suffices to see that in a Noetherian integrally closed domain, the least degree of a polynomial as in (1) showing the integral dependence of $x^{n-1}y$ over (x^n, y^n) is exactly n, where x, y are a regular sequence. This easily follows from the fact that no monomial in a regular sequence can be in the ideal generated by other monomials in the same sequence unless it is formally in the ideal.

Since our feeling is that the tight closure of I is much tighter to I than the integral closure, a natural question which arises is whether there is a bound on the least degree of the integral equation of elements in the tight closure of ideals. We conjecture that such a bound exists, at least for excellent rings.

Conjecture 1.5. Let R be an excellent ring containing a field. There exists an integer N, depending only on R, such that for all ideals I and $x \in I^*$, x satisfies an equation, $x^k + a_1x^{k-1} + \cdots + a_k = 0$ with $a_i \in I^i$ and $k \le N$.

Such uniform bound questions played an important role in the proof of the uniform Artin–Rees theorem in [8]. The main theorem of this paper will prove Conjecture 1.5 provided for all minimal primes P, R/P has a resolution of singularities obtained by blowing up an ideal.

A crucial part in the proof is the notion of test elements and what we call a strong test ideal. In general, the notion of a test element plays an ubiquitous and important part of tight closure theory. The element c in the definition of tight closure can apparently vary with I and x. However for many rings, there are elements which can be used in *all* tight closure tests. We call them *test elements*.

Definition 1.6. An element $c \in R^{o}$ such that $cI^* \subseteq I$ for all I is called a *test element*.

In positive characteristic p, observe that $x \in I^*$ implies that $x^q \in (I^{[q]})^*$ for all $q = p^e$. It follows that $cx^q \in I^{[q]}$ for all q if c is a test element and $x \in I^*$.

The existence of such elements plays a crucial role in tight closure theory. The best one could hope for is that if $c \in R^{o}$ is an element such that every ideal of R_c is tightly closed, then c has a power which is a test element. Unfortunately, we are not able to prove this. We do have the following:

Theorem 1.7. *Let R be one of the following type of rings:*

(1) *A reduced algebra of finite type over an excellent local ring of characteristic p.*

(2) *A reduced ring of characteristic p which is F-finite.*

(3) *A reduced ring essentially of finite type over a field of characteristic* 0.

Let $c \in R^o$ be such that R_c is regular. Then c has a power which is a test element for R.

Proof. In positive characteristic, see [6, (6.20)] for the proof. In equicharacteristic 0, see [7].

Let us call a ring R *acceptable* if it satisfies one of the conditions of Theorem 1.7.

The *test ideal* is the ideal in R generated by all test elements. We usually denote it by $\tau = \tau(R)$. For every I, $\tau I^* \subseteq I$. One of the main results of this paper will be to show the existence of an ideal with even a stronger property.

Definition 1.8. An ideal J, not contained in any minimal prime of R, is said to be a *strong test ideal* if for every ideal I, $JI^* = JI$.

A strong test ideal J has a much stronger property that the test ideal τ: not only is $JI^* \subseteq I$, but $JI^* \subseteq JI$ (and therefore is equal to JI). Evidently, a strong test ideal is not unique. For instance, a multiple of it with an arbitrary ideal not in any minimal prime will also be a strong test ideal. On the other hand, there will be a unique largest such ideal, since, evidently, the sum of two such ideals as well as the union of a collection of such ideals satisfies the same condition. It might be preferable to call this unique largest such ideal *the* strong test ideal, but we have elected not to do so. We will prove that a strong test ideal can be chosen whose radical defines the singular locus of R, provided for every minimal prime P of R, R/P has a resolution of singularities given by blowing up an ideal.

Theorem 1.9 (Existence of a strong test ideal). *If R is an acceptable ring such that for every minimal prime P of R, R/P has a resolution of singularities by blowing up an ideal, then there exists a strong test ideal J. Furthermore, J can be chosen in such a way that $V(J)$ defines the singular locus of R.*

Proof. We begin by reducing to the case in which R is a domain.

Lemma 1.10. *Let R be a reduced ring with minimal primes $P_i, 1 \leq i \leq n$. Assume that there is a nonzero strong test ideal J_i in R/P_i for all i. Then R has a nonzero strong test ideal, $I = \sum_i (P_1 \cap \cdots \cap P_{i-1} \cap P_{i+1} \cap \cdots \cap P_n) I_i$, where I_i is J_i lifted back to R. Furthermore, if each J_i defines the locus of nonregular primes in R_i, then I defines the locus of nonregular primes in R.*

Proof. Let K be an ideal in R. Write $(\ )_i$ to denote images in $R/P_i = R_i$. We know from Proposition 1.3(2) that $(K^*)_i \subseteq (K_i)^*$, so that by definition $J_i(K^*)_i \subseteq J_iK_i$. Lifting this equation back to R we find that for all i,

$$I_iK^* \subseteq I_iK + P_i.$$

Multiplying by $Q_i = (P_1 \cap \cdots \cap P_{i-1} \cap P_{i+1} \cap \cdots \cap P_n)$ yields that $Q_iI_iK^* \subseteq Q_iI_iK$ for all i. It then follows that $IK^* \subseteq IK$, which proves that I is a strong test ideal. If I is contained in any minimal prime P_k, then one obtains that $I_kQ_k \subseteq P_k$, and therefore $I_k \subseteq P_k$. Since the image of I_k in R_k is nonzero this is a contradiction.

It remains to prove the last claim. Write S_{sing} for the set of primes P in $\operatorname{Spec}(S)$ such that S_P is not regular. We need to prove that $V(I) = R_{\text{sing}}$ if $V(J_i) = (R_i)_{\text{sing}}$ for each i. Suppose that $I \subseteq P$. If R_P is regular, then it is a domain. Let $P_i \subseteq P$. Then $(P_i)_P = 0$, so that $R_P \cong (R_i)_P$. Then P cannot contain I_i. But P contains $(P_1 \cap \cdots \cap P_{i-1} \cap P_{i+1} \cap \cdots \cap P_n)I_i$, and the fact that P_i becomes zero after localizing at P implies that the annihilator of P_i, which is $(P_1 \cap \cdots \cap P_{i-1} \cap P_{i+1} \cap \cdots \cap P_n)$ is not contained in P. This forces $I_i \subseteq P$, and thus R_P is not regular. It follows that $V(I) \subseteq R_{\text{sing}}$.

Conversely suppose that R_P is not regular, but that $I \nsubseteq P$. Then there exists an i such that P does not contain $(P_1 \cap \cdots \cap P_{i-1} \cap P_{i+1} \cap \cdots \cap P_n)I_i$. Since P does not contain $(P_1 \cap \cdots \cap P_{i-1} \cap P_{i+1} \cap \cdots \cap P_n)$ it follows that $R_P \cong (R_i)_P$. As P does not contain I_i, it holds that $(R_i)_P$ is regular. □

We now finish the proof of Theorem 1.9. Using Lemma 1.10 we can assume that R is an acceptable domain which has a resolution of singularities by blowing up an ideal I. We claim there is a power of I which is a strong test ideal. Note that I will define the singular locus of R since its blowup gives a resolution of singularities. Let $S = R[It]$ be the Rees algebra of I, so that $X = \operatorname{Proj}(S)$ is a resolution of singularities of $\operatorname{Spec}(R)$. Let $y \in I$ and consider the ring S_{yt}. We see that $S_{yt} \cong R[I/y][yt,(yt)^{-1}]$ is regular. It follows from Theorem 1.7 that there is a power of yt which is a test element. We may do this for each generator y of I, and conclude eventually that some power of ItS consists of test elements for S. Fix this power N. We claim that for all ideals J in R, $I^NJ^* = I^NJ$. Let J be an ideal of R and let $x \in J^*$. Evidently, $x \in (JS)^*$ and so $I^Nt^Nx \subseteq JS$. But I^Nt^Nx lives in degree N in S, so we must have that $I^Nt^Nx \subseteq JS_N$, where S_N represents the Nth graded piece of S. Since $S_N = I^Nt^N$, the claim follows as does the theorem. □

Example 1.11. One example of a strong test ideal was provided by Janet Cowden [1]. Let (R,m) be a one-dimensional complete local domain, and let C be the conductor ideal. In [1] it is noted that C is a strong test ideal. Let I be an arbitrary ideal of R. The tight closure of I is simply the integral closure of I. But the integral closure of I is $IS \cap R$, where S is the integral closure of R. Hence for all ideals I of R, $CI^* = C(IS \cap R) \subseteq CSI = CI$. It is also easy to see in this case that the conductor is the entire test ideal, so that the test ideal is exactly the largest possible strong test ideal.

2. Applications

We begin by showing that Conjecture 1.5 has an affirmative answer when R has a strong test ideal.

Theorem 2.1. *Let R be a Noetherian ring such that R/P has a nonzero strong test ideal for every minimal prime P of R. Then there exists an integer M with the following property: for every ideal I and every $x \in I^*$, there is an equation*

$$x^n + a_1x^{n-1} + \cdots + a_n = 0 \tag{2}$$

with $a_i \in I^i$ and with $n \leq M$.

Proof. We can first go modulo the nilradical N; if $x \in I^*$, then the image of x in R/N is in the tight closure of the image of I in R/N. Suppose we prove Theorem 2.1 for R/N with a uniform bound M. Then there will be an equation also in R of degree Mk where $N^k = 0$. Namely, suppose that $x^n + a_1x^{n-1} + \cdots + a_n \in N$ with $a_i \in I^i$. Raising to the kth power gives an equation as in (2) of degree Mk.

We may assume that R is reduced. Suppose that the minimal primes of R are $P_1, \ldots, P_j$ and we have shown the theorem in R/P_i for each $1 \leq i \leq j$ with uniform bounds M_i. Let $M = M_1 + \cdots + M_j$, and let $x \in I^*$. Write $(\ )_i$ for images in $R_i = R/P_i$. We know that $x_i \in I_i^*$ for each i and then by assumption, there is an equation $f_i(x_i) = 0$, where $\deg(f_i) \leq M_i$ and $f_i(X) = X^{n_i} + a_{1i}X^{n_i-1} + \cdots + a_{n_ii}$ with $a_{ki} \in I_i^k$. Lift the coefficients of f_i back to R. Call the resulting polynomial F_i, and set $F = \prod_{1 \leq i \leq j} F_i$. The degree of F is at most M, $F(x) = 0$, and $F(X)$ has the required form (2).

The case in which R is a domain remains. In this case let J be a nonzero strong test ideal and set N equal to the number of generators of J. If $x \in I^*$, then $Jx \subseteq JI$ and then the determinant trick proves that x satisfies an integral equation over I of the form (2) of degree N. Specifically, choose generators $y_1, \ldots, y_d$ for J, and write equations

$$y_ix = \sum_j a_{ij}y_j$$

with $a_{ij} \in I$. These equations give rise to a matrix equation. Let

$$A = \begin{pmatrix} x - a_{11} & -a_{12} & \cdots & -a_{1d} \\ \vdots & \vdots & \cdots & \vdots \\ -a_{d1} & -a_{d2} & \cdots & x - a_{dd} \end{pmatrix}.$$

Then

$$A \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix} = 0.$$

It then follows that the determinant of A kills all the y_i and hence the determinant is 0. Expanding the determinant yields the desired equation. $\square$

Corollary 2.2. *Let R be an acceptable ring such that R/P has a resolution of singularities by blowing up an ideal for every minimal prime P of R. Then there exists an integer M with the following property: for every ideal I and every $x \in I^*$, there is an equation*

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

with $a_i \in I^i$ and with $n \leq M$.

Proof. Simply put together Theorem 2.1 with Theorem 1.9. □

Corollary 2.3. *Let R be an acceptable ring such that for every minimal prime P of R there is a resolution of singularities for R/P obtained by blowing up an ideal. There exists an integer M, depending only on R, with the following property: if I is an ideal generated by t elements, and $u \in \overline{I^t}$, then there is an equation of integral dependence for u over I of degree at most M.*

Proof. This follows at once from Theorem 1.9 together with the tight closure Briançon–Skoda theorem: $\overline{I^t} \subseteq I^*$. See [2, (5.4)], for positive characteristic and [5, (6.4)] for equicharacteristic 0. □

The next corollary is a quick application of the existence of strong test ideals. It sounds stronger than it seems to be in practice. Often the ideal $\sum_i (y_1, \ldots, \hat{y}_i, \ldots, y_n) \colon y_i$ in Corollary 2.4 will be a maximal ideal.

Corollary 2.4. *Let R be a Noetherian equicharacteristic ring with a strong test ideal $J = (y_1, \ldots, y_n)$. If $\sum_i (y_1, \ldots, \hat{y}_i, \ldots, y_n) \colon y_i \subseteq I$, then $I^* = I$.*

Proof. Let $u \in I^*$. For each i, $1 \leq i \leq n$, we know that $y_i u \in JI$, so we may write $y_i u = \sum_j a_{ij} y_j$ where $a_{ij} \in I$. It follows that $u - a_{ii} \in (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \colon y_i \subseteq I$. Hence $u \in I$. □

3. Questions

The existence of the strong test ideal raises a number of interesting questions for further study.

Question 3.1. Let R be an excellent local domain containing a field. Does there exist a strong test ideal whose radical is equal to the radical of the test ideal?

Question 3.2. In a similar vein as Question 3.1, does there exist a strong test ideal for Cohen–Macaulay tight closure or some variant thereof? For example, let (R, m) be an excellent local ring of arbitrary characteristic. Does there exist an ideal J which defines the non-Cohen–Macaulay locus of R such that for every system of parameters $x_1, \ldots, x_d$,

$$J((x_1, \ldots, x_i) \colon x_{i+1}) = J(x_1, \ldots, x_i)$$

for all $0 \le i \le d-1$ and

$$J((x_1^t, \ldots, x_i^t) \colon (x_1 \cdots x_i)^{t-1}) = J(x_1, \ldots, x_i)$$

for all $1 \le i \le d$?

Question 3.3. Suppose that J is a strong test ideal. Blowing up J gives us a projective scheme over Spec(R) whose affine charts have the property that any element in the tight closure of an ideal I from R is in I expanded to the affine piece of the blowup. This is hardly surprising since we constructed a strong test ideal from a resolution of singularities. Is the blowup of the largest strong test ideal weakly F-regular? (This means that all ideals are tightly closed.) This would be interesting, since such schemes are then F-rational and hence are pseudo-rational [11]. Constructing a blowup with only pseudo-rational singularities would be an important step in understanding resolution of singularities. This means that it would be important to try to construct a strong test ideal independent of resolution of singularities. Its blowup might be pseudo-rational.

Question 3.4. It is reasonable to hypothesize that the largest strong test ideal is tightly closed. The construction of a strong test ideal in this paper will normally give a strong test ideal which is integrally closed. Perhaps the largest strong test ideal is always integrally closed, but it seems more reasonable to ask whether it is tightly closed.

References

[1] J. Cowden, Developing thesis, UCLA, 1997.
[2] M. Hochster, C. Huneke, Tight closure, invariant theory, and the Briançon–Skoda theorem, J. Amer. Math. Soc. 3 (1990) 31–116.
[3] M. Hochster, Solid closure, Contemp. Math. 159 (1994) 103–172.
[4] M. Hochster, Tight closure in equicharacteristic, big Cohen–Macaulay algebras and solid closure, Contemp. Math. 159 (1994) 173–196.
[5] M. Hochster, The notion of tight closure in equal characteristic zero, an Appendix to Tight Closure and its Applications, in: C. Huneke (Ed.), CBMS Regional Conf. Ser., AMS 88 (1996) 94–106.
[6] M. Hochster, C. Huneke, F-regularity, test elements, and smooth base change, Trans. Amer. Math. Soc. 346 (1994) 1–62.
[7] M. Hochster, C. Huneke, Tight closure in equal characteristic zero, in preparation.
[8] C. Huneke, Uniform bounds in Noetherian rings, Invent. Math. 107 (1992) 203–223.
[9] C. Huneke, Tight Closure and its Applications, CBMS Lecture Notes in Mathematics, Vol. 88, American Math. Soc., Providence, RI, 1996.
[10] H. Matsumura, Commutative Algebra, Benjamin, New York, 1970.
[11] K. Smith, F-rational rings have rational singularities, Amer. J. Math., to appear.